REMARK —In the *Mathematical Diary* for 1831, on page 186, it is stated that M. Pagliani had published his solution to this problem, in this last case, when the number of terms is 1000, in the "Annales de Mathematiques" by M. Gergonne.

In the *Mathematical Miscellany* for 1839, on page 127, William Lenhart has given a general solution of this problem, and as a particular case has obtained the same 1000 terms, as were given by M. Pagliani, and I wish, here, to express my high estimation of Mr. Lenhart's valuable contributions to this particular department of mathematics, given in the pages of the *Mathematical Miscellany*.

PROBLEM RELATING TO THE DETERMINATION OF CIRCULAR ORBITS.

BY G. W. HILL, ESQ., NYACK TURNPIKE, N. Y.

Determine the elements of the orbit of a planet or satellite, which moves in a circle in the plane of the ecliptic, from three observations of its direction from the earth, made at equal intervals of time; the positions of the earth and the central body at these times being known, but the sum of the masses of the central body and the planet or satellite being unknown.

Or, geometrically stated,—

In a plane, given a point as center and three straight lines, required to describe a circle, so that the arcs intercepted by the lines taken in a determinate order may be equal.

SOLUTION.

Let generally R denote the sun's distance from the earth,
" " L its longitude,
" " r the constant radius vector of the planet,
" " χ its heliocentric longitude,
" " η its heliocentric angular motion from one observation to the next,
" " λ its longitude as seen from the earth,
" " $\triangle$ its distance from the earth.

Moreover, employ the subscribts $(_{-1})$, $(_0)$, $(_1)$, to denote the special values of the above quantities, which have place at the three times of observation in their order.

By the theory of the transformation of rectangular co-ordinates from

the center of the sun as origin to the center of the earth, we shall have generally the two equations

$$\triangle \cos \lambda = r \cos \chi + R \cos L,$$
$$\triangle \sin \lambda = r \sin \chi + R \sin L.$$

From which may be derived the two

$$\triangle \cos (\lambda - P) = r \cos (\chi - P) + R \cos (L - P),$$
$$\triangle \sin (\lambda - P) = r \sin (\chi - P) + R \sin (L - P),$$

where P is any arbitrary angle. If we apply our equations to each of the three observations, we shall have the six equations

$$\begin{aligned}
\triangle_{-1} \cos \lambda_{-1} &= r \cos (\chi_0 - \eta) + R_{-1} \cos L_{-1},\\
\triangle_{-1} \sin \lambda_{-1} &= r \sin (\chi_0 - \eta) + R_{-1} \sin L_{-1},\\
\triangle_0 \cos \lambda_0 &= r \cos \chi_0 + R_0 \cos L_0,\\
\triangle_0 \sin \lambda_0 &= r \sin \chi_0 + R_0 \sin L_0,\\
\triangle_1 \cos \lambda_1 &= r \cos (\chi_0 + \eta) + R_1 \cos L_1,\\
\triangle_1 \sin \lambda_1 &= r \sin (\chi_0 + \eta) + R_1 \sin L_1.
\end{aligned}$$

These equations contain the six unknowns $\triangle_{-1}$, $\triangle_0$, $\triangle_1$, r, χ and η. If we eliminate $\triangle_{-1}$, $\triangle_0$, $\triangle_1$ from them, we shall have the three equations from which I started in my first solution.* But by retaining $\triangle_0$ as the unknown, we shall arrive at an elegant solution. Let us first then eliminate $\triangle_{-1}$ and $\triangle_1$; this we do by putting $P = \lambda_{-1}$ for the first two equations and $P = \lambda_1$ for the last two. Our equations, for determining the four remaining unknowns, are

$$\begin{aligned}
0 &= r \sin (\chi_0 - \eta - \lambda_{-1}) + R_{-1} \sin (L_{-1} - \lambda_{-1}),\\
\triangle_0 \cos \lambda_0 &= r \cos \chi_0 + R_0 \cos L_0,\\
\triangle_0 \sin \lambda_0 &= r \sin \chi_0 + R_0 \sin L_0,\\
0 &= r \sin (\chi_0 + \eta - \lambda_1) + R_1 \sin (L_1 - \lambda_1).
\end{aligned}$$

If in the second and third of these equations we put successively $P = \eta + \lambda_{-1}$ and $P = -\eta + \lambda_1$, we get

$$\triangle_0 \sin (\lambda_0 - \eta - \lambda_{-1}) = r \sin (\chi_0 - \eta - \lambda_{-1}) + R_0 \sin (L_0 - \eta - \lambda_{-1}),$$
$$\triangle_0 \sin (\lambda_0 + \eta - \lambda_1 = r \sin (\chi_0 + \eta - \lambda_1) + R_0 \sin (L_0 + \eta - \lambda_1).$$

If from these equations we subtract the first and last of the preceding four, we get

$$\triangle_0 \sin (\lambda_0 - \eta - \lambda_{-1}) = R_0 \sin (L_0 - \eta - \lambda_{-1}) - R_{-1} \sin (L_{-1} - \lambda_{-1}),$$
$$\triangle_0 \sin (\lambda_0 + \eta - \lambda_1) = R_0 \sin (L_0 + \eta - \lambda_1) - R_1 \sin (L_1 - \lambda_1).$$

Behold us then, as the French say, arrived at two equations with two

*Mr. Hill here refers to a solution of this question communicated to Dr. Wright, and published in the *Yates County Chronicle* of February 5, 1874.—Ed.

unknowns, and that without complicating the form of our original equations.

It is very easy to eliminate $\triangle_0$ from these, and we get

$$[R_0 \sin (L_0 - \eta - \lambda_{-1}) - R_{-1} \sin (L_{-1} - \lambda_{-1})] \sin (\lambda_0 + \eta - \lambda_1)$$
$$= [R_0 \sin (L_0 + \eta - \lambda_1) - R_1 \sin (L_1 - \lambda_1)] \sin (\lambda_0 - \eta - \lambda_{-1}).$$

But we prefer to keep $\triangle_0$ as our final unknown. Let us put for the sake of brevity

$$\eta = \sigma + \frac{\lambda_1 - \lambda_{-1}}{2}, \quad \delta = \lambda_0 - \frac{\lambda_1 + \lambda_{-1}}{2}, \quad \delta^1 = L_0 - \frac{\lambda_1 + \lambda_{-1}}{2},$$
$$\psi_{-1} = L_{-1} - \lambda_{-1}, \quad \psi_1 = L_1 - \lambda_1.$$

All these are known quantities with the exception of σ, which will take the place of η as an unknown. Our two equations can now be written

$$\triangle_0 \sin (\delta - \sigma) = R_0 \sin (\delta^1 - \sigma) + R_{-1} \sin \psi_{-1},$$
$$\triangle_0 \sin (\delta + \sigma) = R_0 \sin (\delta^1 + \sigma) + R_1 \sin \psi_1.$$

Or by taking in succession the half the sum and half the difference

$$\triangle_0 \sin \delta \cos \sigma = R_0 \sin \delta^1 \cos \sigma + \frac{R_1 \sin \psi_1 + R_{-1} \sin \psi_{-1}}{2},$$

$$\triangle_0 \cos \delta \sin \sigma = R_0 \cos \delta^1 \sin \sigma + \frac{R_1 \sin \psi_1 - R_{-1} \sin \psi_{-1}}{2}.$$

$$\text{Whence } \cos \sigma = \tfrac{1}{2} \frac{R_1 \sin \psi_1 + R_{-1} \sin \psi_{-1}}{\triangle_0 \sin \delta - R_0 \sin \delta^1},$$

$$\sin \sigma = \tfrac{1}{2} \frac{R_1 \sin \psi_1 - R_{-1} \sin \psi_{-1}}{\triangle_0 \cos \delta - R_0 \cos \delta^1}.$$

By putting, (these are all known quantities)

$$a = \frac{R_1 \sin \psi_1 + R_{-1} \sin \psi_{-1}}{2 \sin \delta}, \quad b = \frac{R_1 \sin \psi_1 - R_{-1} \sin \psi_{-1}}{2 \cos \delta},$$

$$c = R_0 \frac{\sin \delta^1}{\sin \delta}, \quad d = R_0 \frac{\cos \delta^1}{\cos \delta},$$

we shall obtain the very elegant form for our final equation determining $\triangle_0$,

$$\left\{ \frac{a}{\triangle_0 - c} \right\}^2 + \left\{ \frac{b}{\triangle_0 - d} \right\}^2 = 1.$$

This is, as we see, of the fourth degree in $\triangle_0$; in the form in which

Problem 400† is stated, this equation will be found to have a root $\triangle_0 = 0$, that is the absolute term of the equation will be 0; in this case therefore the equation reduces to the third degree.

By the introductiou of the new unknown

$$x = \triangle_0 - \frac{c+d}{2}, \text{ and putting } h = \frac{c-d}{2}$$

the equation takes the somewhat simpler form

$$\left\{ \frac{a}{x+h} \right\}^2 + \left\{ \frac{b}{x-h} \right\}^2 = 1,$$

$$\text{or } (x^2 - h^2)^2 = a^2(x-h)^2 + b^2(x+h)^2.$$

SOLUTIONS OF PROBLEMS IN NO. 1.

In this department we will in geueral publish but one solution to each problem proposed, though, in some cases, when the method pursued in the solutions is essentially different, two or more solutions of the same question will be published. Credit will be given, however, in each number to all who shall have furnished correct solutions of the questions whose solutions are published in that number.

Each solution should contain sufficient detail to be comprehended by the ordinary reader who is acquainted with the elements of the branches employed in the solution. Clearness must not be sacrificed to brevity, but, other things being equal, the brevity of a solution will determine its selection for publication.

When several persons furnish essentially the same solution to a question, his name only will be placed at the head of the published solution whose notation and phraseology are adopted.

Persons sending solutions are requested to put the solution of *each question*, together with the name of the writer, on a separate piece of paper.

Solutions have been received as follows:

R. M. DeFrance solved 1 and 2; Theo. L. DeLand solved 1; Prof. A. B. Evans solved 1, 2, 3 and 4; Prof. C. Hornung solved 2; Philip Hoagland solved 2; Henry Heaton solved 1, 2, 3 and 4; Prof. E. W. Hyde solved 3; Prof. Knisely solved 1; Miss Esther W. Matthews, (State Normal School, Kirksville, Mo.), solved 1; Artemas Martin solved 1, 2, 3 and

†*Yates County Chronicle.*